Available online at www.sciencedirect.com

SciVerse ScienceDirect

Journal of Economic Theory 148 (2013) 1953–1976

JOURNAL OF Economic Theory

www.elsevier.com/locate/jet

On Ramsey's conjecture

Tapan Mitra[a], Gerhard Sorger[b,∗,1]

[a] *Department of Economics, Cornell University, Ithaca NY, USA*
[b] *Department of Economics, University of Vienna, Vienna, Austria*

Received 29 January 2013; final version received 19 April 2013; accepted 15 May 2013

Available online 23 May 2013

Abstract

Studying a one-sector economy populated by finitely many heterogeneous households that are subject to no-borrowing constraints, we confirm a conjecture by Frank P. Ramsey according to which, in the long run, society would be divided into the set of patient households who own the entire capital stock and impatient ones without any physical wealth. More specifically, we prove (i) that there exists a unique steady state equilibrium that is globally asymptotically stable and (ii) that along every equilibrium the most patient household owns the entire capital of the economy after some finite time. Furthermore, we prove that despite the presence of the no-borrowing constraints all equilibria are efficient. Our results are derived for the continuous-time formulation of the model that was originally used by Ramsey, and they stand in stark contrast to results that – over the last three decades – have been found in the discrete-time version of the model.

JEL classification: D61; D91; E21; O41

Keywords: Ramsey equilibrium; Ramsey conjecture; Continuous-time formulation; Turnpike property; Global asymptotic stability; Efficiency

∗ Corresponding author.
E-mail addresses: tm19@cornell.edu (T. Mitra), gerhard.sorger@univie.ac.at (G. Sorger).

1 Gerhard Sorger acknowledges financial support by the Austrian Science Fund (FWF): DK-plus W1231.

http://dx.doi.org/10.1016/j.jet.2013.05.003

1. Introduction

Frank Ramsey's seminal paper on "A Mathematical Theory of Saving" from 1928, which forms a corner stone of modern economic growth theory, ends with a conjecture about the distribution of consumption levels in a society consisting of individuals with heterogeneous time-preference rates. More specifically, Ramsey concluded his paper by writing that "equilibrium would be attained by a division of society into two classes, the thrifty enjoying bliss and the improvident at the subsistence level" [14, p. 559]. Although the arguments that Ramsey used in support of his conjecture apply only to constant (i.e., time-independent) equilibrium paths of consumption and wealth, the Ramsey conjecture is nowadays usually interpreted as a "folk result" about the *eventual* or *asymptotic* distribution of wealth in a heterogeneous society. Becker [1] confirmed the conjecture in its original form regarding constant equilibrium paths by first adding a non-negativity constraint on the capital holdings of the households and then proving that this model admits a unique constant equilibrium, in which the most patient household owns the entire wealth of the economy and all other households consume exactly their wage income. At about the same time Bewley [7] proved that in an economy with complete markets, i.e., without the no-borrowing constraint introduced by Becker [1], the consumption levels of all but the most patient households are zero after some finite time in every equilibrium, thereby establishing a link between dynamic general equilibrium theory and turnpike theory. As noted by Becker [2], an unsatisfactory feature of these dynamic equilibria is that the impatient households have zero consumption after a finite time, but continue to provide labor services. This aspect can be avoided with the incomplete market structure of the model in Becker [1], where the no-borrowing constraint means that households can always consume their wage. However, in the context of this model, the asymptotic result on the distribution of wealth fails to hold (for non-stationary equilibria), as has been shown in several papers, beginning with Becker and Foias [4].

It is interesting to note that, while Ramsey formulated his model in a continuous-time framework, both Becker [1] and Bewley [7] used a discrete-time formulation. The main purpose of the present paper is to reconsider the model from Becker [1] in the *continuous-time formulation* originally used by Ramsey [14] and to confirm in that model the strong version of Ramsey's conjecture, i.e., the "folk result" about the eventual distribution of wealth.

In what follows, we shall refer to the dynamic general equilibrium model of Becker [1], which describes a competitive one-sector economy with heterogeneous households that are subject to no-borrowing constraints, as the Ramsey model, and to the equilibria of that model as Ramsey equilibria. The literature about this model up to 2005 has been comprehensively surveyed by Becker [2], who also describes the relation of Ramsey's [14] work to the earlier writings of Rae [13] and Fisher [10] and who discusses in general why knowing the long-run distribution of capital is interesting in models where individuals' time-preference rates differ from each other. We shall therefore only point out those articles on the Ramsey model that are most closely related to our own paper. As has been mentioned above, Ramsey's conjecture about constant equilibria was confirmed by Becker [1]. Subsequent work by Becker and Foias [4] established that every household except for the most patient one must attain the zero-capital state infinitely often on any interval of the form $[T, +\infty)$. This so-called recurrence property is known to be the only major result about the dynamics of Ramsey equilibria that can be proved under standard assumptions in the discrete-time setting. Indeed, Becker and Foias [4,5] and Sorger [16,17] demonstrated that Ramsey equilibria can display non-convergent (periodic or chaotic) behavior, even if the most patient household owns eventually (i.e., after some finite time) all the capital. An example due to Michael L. Stern [reported in Becker [2]] demonstrates that the limes superior of every household's capital stock can be strictly positive, i.e., the zero-capital state may not even be approached asymptotically by impatient households. Finally, Becker et al. [3] provide an example of a Ramsey equilibrium in which the most patient household reaches the zero-capital position infinitely often on any interval of the form $[T, +\infty)$. To summarize, in the discrete-time version of the model that has been proposed by Becker [1], the "folk result" about the eventual ownership pattern cannot be proved under standard assumptions. Furthermore, it has been shown in Becker et al. [3] that the possible non-convergence of discrete-time Ramsey equilibria to the steady state may also be a cause of inefficiency.

In the present paper we analyze the above mentioned issues in the continuous-time formulation of the model. Such an exercise would we futile if it simply confirmed the results from the discrete-time analysis. It turns out, however, that the continuous-time approach allows both for a more general and for a more precise characterization of the dynamics and the efficiency properties of Ramsey equilibria. It is more general in the sense that certain properties which can be proved in the discrete-time model only under additional (non-standard) assumptions hold in the continuous-time model without such assumptions. And it is more precise in the sense that one can derive monotonicity results about Ramsey equilibria in continuous time that do not necessarily hold in the discrete-time framework.

We start by proving that there exists a unique steady state equilibrium. In this equilibrium the most patient household owns the entire capital stock. Then we show that every equilibrium satisfies the turnpike property, that is, there exists a finite time T such that all households except for the most patient one hold no capital from time T onwards. This fully confirms the strong version of Ramsey's conjecture, i.e., the "folk result" about the eventual capital ownership pattern, in the continuous-time version of Becker's [1] incomplete markets economy. We can also show that the unique steady state equilibrium is globally asymptotically stable, that is, all individual capital holdings and consumption levels, the aggregate capital stock, as well as both factor prices converge along every equilibrium to their respective steady state values. Obviously, this rules out oscillating or chaotic equilibria like those known to exist in the discrete-time model. Moreover, we are able to prove that Ramsey equilibria can be of only two types. Either the aggregate capital stock eventually exceeds its steady state value and the equilibrium converges monotonically towards the unique steady state, or the aggregate capital stock remains eventually below its steady state value. Finally we prove that, in contrast to the discrete-time setting and despite the presence of the no-borrowing constraints, all equilibria in the continuous-time Ramsey model are efficient.

We would like to point out that the structure of the economy studied in the present paper is identical to that analyzed by Becker [1] and his followers mentioned above; the only distinction is indeed the formulation of time. Thus, the drastic differences between the results that have been found for the discrete-time version and those from the present paper cannot be explained by economic intuition. Instead they can be rooted only in different (topological) structures of the solution spaces of difference equations and differential equations, respectively. In this respect it must be emphasized, however, that the equilibrium dynamics of the heterogeneous-agent economy under consideration are described by a high-dimensional system of differential equations. Our arguments are therefore necessarily much more involved than the simple observation that trajectories of one-dimensional differential equations must be monotonic.

The rest of the paper is organized as follows. In Section 2 we formulate the model and state the assumptions which will be maintained throughout the paper. Section 3 presents the main results and relates them to corresponding findings in the discrete-time model. All proofs are collected in Section 4. In the final Section 5 we make a couple of concluding remarks.

2. Model formulation

Time evolves continuously with the time variable t taking values in $\mathbb{R}_+ = [0, +\infty)$. We shall also use the notation $\mathbb{R}_{++} = (0, +\infty)$. The economy is populated by a fixed and finite number H of infinitely-lived households, which own the production factors capital and labor, supply them on the respective factor markets to the firms in the (single) production sector, and use the resulting factor income to buy output. Output can be consumed or saved (i.e., turned into capital). The production sector consists of infinitely many identical firms, which rent the production factors from the households and produce output. All agents in the economy act as price takers. All three markets clear at every instant of time.

2.1. Firms

At every instant $t \in \mathbb{R}_+$ there exists a continuum of measure 1 of identical firms, which have access to a production technology described by the function $F : \mathbb{R}_+^2 \mapsto \mathbb{R}_+$. Here, $F(K, L)$ denotes the amount of output that can be produced with K units of capital and L units of labor. The firms at time t take the current rental rate of capital, $r(t)$, and the current real wage rate, $w(t)$, as given and maximize their profit

$$F\big(K(t), L(t)\big) - r(t)K(t) - w(t)L(t)$$

with respect to the factor inputs $K(t)$ and $L(t)$.

The production function F satisfies the usual neoclassical assumptions including continuity and linear homogeneity. We define the function $f : \mathbb{R}_+ \mapsto \mathbb{R}_+$ by $f(K) = F(K, H)$, where H is the number of households; see below. It is assumed that f is continuous on $\mathbb{R}_+$ and twice continuously differentiable on $\mathbb{R}_{++}$ with $f(0) = 0$, $f'(K) > 0$, and $f''(K) < 0$ for all $K \in \mathbb{R}_{++}$. Furthermore, we assume that the Inada conditions $\lim_{K\to 0} f'(K) = +\infty$ and $\lim_{K\to+\infty} f'(K) = 0$ hold.

The function $W : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is defined by $W(0) = 0$ and $W(K) = [f(K) - Kf'(K)]/H$ for all $K \in \mathbb{R}_{++}$. The assumptions on f imply that W is differentiable on $\mathbb{R}_{++}$ with $W(K) > 0$ and $W'(K) > 0$ for all $K \in \mathbb{R}_{++}$. Note that the above definitions imply that $(\partial/\partial K)F(K, H) = f'(K)$ and $(\partial/\partial L)F(K, H) = W(K)$ for all $K \in \mathbb{R}_{++}$.

2.2. Households

There exist $H \in \mathbb{N}$ households indexed by $h \in \mathcal{H} := \{1, 2, \dots, H\}$. Each household lives throughout the entire time-domain $\mathbb{R}_+$ and is specified by a triple (u^h, ρ^h, k_0^h), where $u^h : \mathbb{R}_+ \mapsto \mathbb{R}$ is the utility function, $\rho^h > 0$ is the time-preference rate, and $k_0^h \geqslant 0$ is the initial endowment of capital. It is assumed that the aggregate capital endowment of the economy, $K_0 := \sum_{h=1}^{H} k_0^h$, is strictly positive. Furthermore, each household h is endowed with a constant flow of labor normalized to 1.[2]

All households act as price takers and have perfect foresight. More specifically, the households take the entire time paths of the rental rate of capital, $r : \mathbb{R}_+ \mapsto \mathbb{R}_+$, and the wage rate, $w : \mathbb{R}_+ \mapsto \mathbb{R}_+$, as given. These time paths are continuous. Household h seeks to maximize the objective functional

[2] Allowing heterogeneity with respect to the labor endowments does not add anything interesting.

$$\int_0^{+\infty} e^{-\rho^h t} u^h\big(c^h(t)\big)\,\mathrm{d}t$$

subject to the flow budget constraint

$$\dot{k}^h(t) = \big[r(t) - \delta\big]k^h(t) + w(t) - c^h(t), \tag{1}$$

the no-borrowing constraint

$$k^h(t) \geqslant 0, \tag{2}$$

the initial condition $k^h(0) = k_0^h$, and the non-negativity constraint on consumption

$$c^h(t) \geqslant 0. \tag{3}$$

Here, $c^h(t)$ is the consumption rate at time t, $k^h(t)$ denotes the capital holdings at time t, and $\delta > 0$ is the rate of capital depreciation. A pair (k^h, c^h) consisting of a capital path and a consumption path is *feasible*, if k^h is continuous and piecewise differentiable, if c^h is piecewise continuous with finite left-hand and right-hand limits, if $k^h(0) = k_0^h$, if the non-negativity constraints (2) and (3) hold for all $t \in \mathbb{R}_+$, and if the state equation (1) holds for all $t \in \mathbb{R}$ at which k^h is differentiable.[3]

We assume that, for all $h \in \mathcal{H}$, the utility function $u^h : \mathbb{R}_+ \mapsto \mathbb{R}$ is continuous on $\mathbb{R}_+$ and twice differentiable on $\mathbb{R}_{++}$. Furthermore, we assume that $(u^h)'(c^h) > 0$ and $(u^h)''(c^h) < 0$ hold for all $c^h \in \mathbb{R}_{++}$ and that $\lim_{c^h \to 0}(u^h)'(c^h) = +\infty$.

Finally, we assume that there exists a unique most patient household, and we order the households according to increasing impatience, that is, $0 < \rho^1 < \rho^2 \leqslant \rho^3 \leqslant \cdots \leqslant \rho^H$.

2.3. Market clearing

The labor market clears at time t if

$$L(t) = H, \tag{4}$$

the capital market clears at time t if

$$K(t) = \sum_{h=1}^{H} k^h(t), \tag{5}$$

and the output market clears at time t if

$$\dot{K}(t) + \delta K(t) + \sum_{h=1}^{H} c^h(t) = f\big(K(t)\big). \tag{6}$$

In all three market clearing equations, the left-hand side denotes the demand whereas the right-hand side is the supply.

---

[3] These regularity assumptions are usually made in optimal control models in economics; see Seierstad and Sydsæter [15].

2.4. Equilibrium

An equilibrium for the economy described by the production function F and the households' characteristics $\{(u^h, \rho^h, k_0^h) \mid h \in \mathcal{H}\}$ is a family of real-valued functions $(K, L, r, w, \{k^h, c^h \mid h \in \mathcal{H}\})$ defined on the common domain $\mathbb{R}_+$ such that the following conditions hold:

(i) Given the price paths (r, w) it holds for all $h \in \mathcal{H}$ that the individual allocation (k^h, c^h) solves the utility maximization problem of household h.
(ii) For all $t \in \mathbb{R}_+$ and given the prices $(r(t), w(t))$ it holds that the aggregate allocation $(K(t), L(t))$ solves the firms' profit maximization problem.
(iii) All markets clear at all times $t \in \mathbb{R}_+$.

3. Results

In this section we present our results and compare or contrast them to related findings in the discrete-time Ramsey model. All proofs can be found in Section 4.

An equilibrium is called a steady state equilibrium, if it consists of functions that are constant with respect to time. The following theorem proves that there exists a unique steady state equilibrium. This result is the continuous-time counterpart to the main theorem in Becker [1]. To state the result we introduce the notation $r^* = \rho^1 + \delta$ and we define the values $K^* \in \mathbb{R}_{++}$ and $w^* \in \mathbb{R}_{++}$ by $f'(K^*) = r^*$ and $w^* = W(K^*)$, respectively.

Theorem 1. *There exists a unique steady state equilibrium with a positive aggregate capital stock. In this equilibrium it holds that* $r(t) = r^*$, $w(t) = w^*$, $K(t) = k^1(t) = K^*$, $c^1(t) = (r^* - \delta)K^* + w^*$, *as well as* $k^h(t) = 0$ *and* $c^h(t) = w^*$ *for all* $h \geqslant 2$ *and all* $t \in \mathbb{R}_+$.

The steady state equilibrium features a degenerate wealth distribution in which only the most patient household owns any capital whereas the less patient ones live off their wage incomes. Theorem 1 therefore confirms Ramsey's conjecture about the wealth distribution in a constant equilibrium. However, an important open question is whether the steady state equilibrium is in some sense stable, that is, whether all equilibria approach the steady state over time. There are at least two ways in which one can interpret this question. First, convergence could mean that the wealth distribution becomes degenerate and, second, it could mean that the factor prices, capital holdings, and consumption rates converge to their respective steady state values. We shall now show that both of these properties hold in the present model.

One of the weakest convergence properties of the first type is the recurrence property. An equilibrium is said to satisfy this property, if for every household $h \geqslant 2$ there exists a sequence of time instants $(t_i^h)_{i=1}^{+\infty}$ such that $\lim_{i \to +\infty} t_i^h = +\infty$ and $k^h(t_i^h) = 0$ for all $i \in \mathbb{N}$. The recurrence property therefore says that all households except for the most patient one possess no capital infinitely often on any interval of the form $[T, +\infty)$ with $T \in \mathbb{R}_+$. For the discrete-time model, Becker and Foias [4] have proved that every equilibrium satisfies the recurrence property, and Becker [2, p. 427] has noted that "the recurrence theorem is the most general result in the literature on the properties in a dynamic Ramsey equilibrium". Indeed, an example due to Michael L. Stern [reported in Becker [2]] demonstrates that the zero-capital state may not even be approached asymptotically by the impatient households, and another example due to Becker et al. [3] shows that the most patient household may reach the zero-capital position infinitely often on any interval of the form $[T, +\infty)$. Thus, the recurrence property does not confirm the "folk result" about the eventual distribution of wealth.[4]

The "folk result" would only be established if it were true that the most patient household owns the entire capital stock from some finite time onwards. Becker and Foias [4] call this the turnpike property and define it formally in the following way: an equilibrium satisfies the turnpike property, if there exists $T \in \mathbb{R}_+$ such that $K(t) = k^1(t)$ and $k^h(t) = 0$ hold for all $h \geqslant 2$ and all $t \geqslant T$.

Theorem 2. *Every equilibrium satisfies the turnpike property.*

For the discrete-time version of the model it is known that the turnpike property does not hold for all equilibria unless additional non-standard assumptions are imposed on the form of the production function. Furthermore, it was shown by Becker and Foias [4,5] and Sorger [16,17] that, even when an equilibrium in the discrete-time model satisfies the turnpike property, it need not converge to the steady state equilibrium. As a matter of fact, these authors have constructed periodic equilibria, chaotic equilibria, and even sunspot equilibria of the discrete-time Ramsey model which satisfy the turnpike property. In the continuous-time model, on the other hand, the unique steady state equilibrium is globally asymptotically stable, which rules out any form of complicated equilibrium dynamics. Here, global asymptotic stability of the steady state equilibrium is defined in the sense that the aggregate capital stock $K(t)$, both factor prices $r(t)$ and $w(t)$, all individual capital holdings $k^h(t)$, as well as all individual consumption rates $c^h(t)$ converge to their respective steady state values as t approaches infinity.

Theorem 3. *The unique steady state equilibrium is globally asymptotically stable.*

Theorem 3 goes beyond the "folk result" by showing that, in addition to the wealth distribution becoming degenerate, all variables of the model converge asymptotically towards their steady state values. As a matter of fact, one can derive even more properties of the equilibrium dynamics in the continuous-time model, some of which are stated in the following theorem. A corresponding result for the discrete-time model is known under additional (non-standard) assumptions on the production function [see Becker et al. [3]] or in a variant of the discrete-time model in which wages are paid out of capital rather than out of output [see Borissov [8]].

Theorem 4. *There exists $T \in \mathbb{R}_+$ such that one of the following two statements is correct*:

(a) *It holds that $K(t) \geqslant K^*$ for all $t \geqslant T$ and K is non-increasing on $[T, +\infty)$.*
(b) *It holds that $K(t) \leqslant K^*$ for all $t \geqslant T$.*

We conclude this section by stating a result on efficiency. Given the aggregate capital endowment $K_0 = \sum_{h=1}^{H} k_0^h$, an aggregate capital path $K : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is feasible if $K(0) = K_0$ and if $f(K(t)) - \delta K(t) - \dot{K}(t) \geqslant 0$ holds for all $t \in \mathbb{R}_+$ at which K is differentiable. The aggregate consumption path corresponding to that aggregate capital path is given by $C(t) =$

[4] It will be proved in Lemma 8 in Section 4 that the recurrence property holds also for all continuous-time Ramsey equilibria. We do not state this result in the present section because Theorem 2 below establishes a much stronger property.

$f(K(t)) - \delta K(t) - \dot{K}(t)$. An equilibrium with aggregate consumption path $C(t)$ is efficient, if there exists no feasible aggregate capital path $\tilde{K}$ emanating from K_0 with associated aggregate consumption path $\tilde{C}$ such that $\tilde{C}(t) \geqslant C(t)$ holds for all $t \in \mathbb{R}_+$ and $\tilde{C}(t) > C(t)$ holds for all t in a subset of $\mathbb{R}_+$ that has positive Lebesgue measure. In the discrete-time model it has been shown by Becker et al. [3] that not all equilibria are efficient; see also Becker and Mitra [6]. Our last result demonstrates that all equilibria in the continuous-time Ramsey model are efficient.

Theorem 5. *Every equilibrium is efficient.*

4. Proofs

In this section we present the proofs of all theorems stated in Section 3. We shall also outline which intermediate results hold or fail, respectively, in the discrete-time setting.

4.1. Equilibrium conditions

Let us start with the firms' optimization problem at instant t. It is well known that the necessary and sufficient first-order optimality conditions for this problem are given by

$$r(t) = f'\big(K(t)\big) \quad \text{and} \quad w(t) = W\big(K(t)\big). \tag{7}$$

Now let us turn to household h's utility maximization problem, where $h \in \mathcal{H}$. We denote by μ^h and ν^h the adjoint variable corresponding to the budget constraint (1) and the Lagrange multiplier corresponding to the no-borrowing constraint (2), respectively. The first-order optimality conditions of the maximum principle for the utility maximization problem of household h can be stated as follows; see Hartl et al. [11, Theorem 4.1] or Feichtinger and Hartl [9, Theorem 6.2]:

$$\big(u^h\big)'\big(c^h(t)\big) = \mu^h(t), \tag{8}$$

$$\dot{\mu}^h(t) = \big[\rho^h + \delta - r(t)\big]\mu^h(t) - \nu^h(t), \tag{9}$$

$$\nu^h(t) \geqslant 0, \tag{10}$$

$$k^h(t)\nu^h(t) = 0. \tag{11}$$

Condition (8) shows that $\mu^h(t)$ equals the marginal utility of consumption and, therefore, it must hold for all $t \in \mathbb{R}_+$ that $\mu^h(t) > 0$.

An interval $I \subseteq \mathbb{R}_+$ such that $k^h(t) = 0$ holds for all $t \in I$ is called a *boundary interval* for household h's optimization problem. Analogously, an interval $I \subseteq \mathbb{R}_+$ such that $k^h(t) > 0$ holds for all $t \in I$ is called an *interior interval*. Because of the continuity of k^h, boundary intervals must be closed and interior ones must be open. A time instant $\bar{t} \in \mathbb{R}_+$, at which the no-borrowing constraint (2) becomes binding (i.e., $k^h(\bar{t} - \varepsilon) > 0$ and $k^h(\bar{t} + \varepsilon) = k^h(\bar{t}) = 0$ for all sufficiently small $\varepsilon > 0$), is called an *entry point*. A time instant $\bar{t} \in \mathbb{R}_+$, at which the constraint ceases to be binding (i.e., $k^h(\bar{t} - \varepsilon) = k^h(\bar{t}) = 0$ and $k^h(\bar{t} + \varepsilon) > 0$ for all sufficiently small $\varepsilon > 0$), is an *exit point*. An isolated time instant $\bar{t} \in \mathbb{R}_+$, at which the constraint is binding (i.e., $k^h(\bar{t}) = 0$, $k^h(\bar{t} - \varepsilon) > 0$, and $k^h(\bar{t} + \varepsilon) > 0$ for all sufficiently small $\varepsilon > 0$), is a *contact point*. Entry points, exit points, and contact points together form the set of *junction points*.

Lemma 1. *For all $h \in \mathcal{H}$ it holds that c^h and μ^h are continuous functions of time and that they are piecewise continuously differentiable with possible kinks only at junction points. The aggregate variables K, r, and w are continuously differentiable.*

Proof. Because the Hamiltonian function of household h's optimization problem,

$$G^h(k^h, c^h, \mu^h, \nu^h, t) = u^h(c^h) + \mu^h\{[r(t) - \delta]k^h + w(t) - c^h\} + \nu^h k^h,$$

is strictly concave with respect to the consumption rate c^h, it follows that the optimal control path c^h is continuous on $\mathbb{R}_+$; see, e.g., Seierstad and Sydsæter [15, p. 86] or Feichtinger and Hartl [9, Corollary 6.2]. Because of condition (8), this implies that the adjoint variable μ^h is also continuous on $\mathbb{R}_+$.

Since all individual capital paths k^h are assumed to be continuous and piecewise differentiable, it follows from (5) that K is continuous and piecewise differentiable. Having shown that all individual consumption paths c^h are continuous it follows from (6) that $\dot{K}$ must be continuous. These observations prove that K is continuously differentiable. Because of (7) the factor prices r and w must also be continuously differentiable.

On boundary intervals it must hold that $c^h(t) = w(t) = W(K(t))$, which together with continuous differentiability of K shows that c^h must be continuously differentiable on the interior of such an interval. Appealing again to condition (8) it follows that μ^h is also continuously differentiable on the interior of a boundary interval. On an interior interval it follows from (9)–(11) that $\dot{\mu}^h(t) = [\rho^h + \delta - r(t)]\mu^h(t)$. This shows that μ^h is continuously differentiable on such an interval and, appealing again to (8), c^h must be continuously differentiable as well. □

Lemma 2. *Let $\bar{t}$ be any junction point for household h's optimization problem. Then it follows that $c^h(\bar{t}) = w(\bar{t})$. If $\bar{t}$ is an entry point or a contact point, then there does not exist $\varepsilon > 0$ such that $c^h(t) \leqslant w(t)$ holds for all $t \in (\bar{t} - \varepsilon, \bar{t})$. Analogously, if $\bar{t}$ is an exit point or a contact point, then there does not exist $\varepsilon > 0$ such that $c^h(t) \geqslant w(t)$ holds for all $t \in (\bar{t}, \bar{t} + \varepsilon)$.*

Proof. On any boundary interval it holds that $k^h(t) = 0$. If such an interval has non-empty interior, then we must obviously have $\dot{k}^h(t) = 0$ on the interior. Substituting this into (1) one obtains $c^h(t) = w(t)$ on the interior of a boundary interval. By continuity of c^h and w this equality holds also for entry and exit points. If $\bar{t}$ is a contact point, it must be a local minimum of k^h. This implies that $k^h(\bar{t}) = \dot{k}^h(\bar{t}) = 0$ and $c^h(\bar{t}) = w(\bar{t})$ follows again from (1).

Now suppose that $\bar{t}$ is an entry or contact point and that there exists $\varepsilon > 0$ such that $c^h(t) \leqslant w(t)$ holds for all $t \in (\bar{t} - \varepsilon, \bar{t})$. It is straightforward to see from (1) and $k^h(\bar{t}) = 0$ that this would imply that $k^h(t) \leqslant 0$ holds for all $t \in (\bar{t} - \varepsilon, \bar{t})$. Since this is a contradiction to $\bar{t}$ being an entry or contact point, there does not exist $\varepsilon > 0$ such that $c^h(t) \leqslant w(t)$ holds for all $t \in (\bar{t} - \varepsilon, \bar{t})$. The statement about exit or contact points can be proved analogously. □

4.2. Proof of Theorem 1

We first show that there can exist at most one steady state equilibrium and that this equilibrium must satisfy the formulas stated in the theorem. In a steady state equilibrium the aggregate capital stock K must be constant, say, $K(t) = \tilde{K}$ for all $t \in \mathbb{R}_+$. Together with condition (7) this implies that $r(t) = \tilde{r} := f'(\tilde{K})$ for all $t \in \mathbb{R}_+$. Constancy of c^h together with (8) implies that μ^h is constant and, hence, $\dot{\mu}^h(t) = 0$ for all $t \in \mathbb{R}_+$. Substituting this together with (10), $\mu^h(t) > 0$, and $r(t) = \tilde{r}$ into (9) it follows that

$$\rho^h + \delta - \tilde{r} \geqslant 0 \tag{12}$$

holds for all $h \in \mathcal{H}$ whereby, due to (11), the equality must hold whenever $k^h(t) > 0$. Now suppose that there exist $h \geqslant 2$ and $t \in \mathbb{R}_+$ such that $k^h(t) > 0$. In this case (12) must hold as equality and we obtain $\rho^1 + \delta - \tilde{r} < \rho^h + \delta - \tilde{r} = 0$, where we have used $\rho^1 < \rho^h$ for all $h \geqslant 2$. Since (12) must also hold for $h = 1$, this is a contradiction. Hence $k^h(t) = 0$ must hold for all $h \geqslant 2$ and all $t \in \mathbb{R}_+$ and, consequently, $k^1(t) = K(t) = \tilde{K} > 0$ holds for all $t \in \mathbb{R}_+$. Appealing again to (12), of which we now know that it must hold as an equality for $h = 1$, we obtain $\tilde{r} = \rho^1 + \delta = r^*$. Together with (7) this implies $\tilde{K} = K^*$. Finally, by substituting the above results into (1) we obtain for all $h \geqslant 2$ and all $t \in \mathbb{R}_+$ that $c^h(t) = w(t) = W(K^*) = w^*$, and by substituting all of these results into (6) it follows that $c^1(t) = f(K^*) - \delta K^* - (H-1)w^* = (r^* - \delta)K^* + w^*$.

We have already mentioned that the conditions in (7) are sufficient for the firms' profit maximization problem. Because of the convexity properties of the model, the conditions stated in (8)–(11) are also sufficient for the households' optimization problems provided that the transversality condition holds. The latter, however, is trivially satisfied along a steady state equilibrium. This shows that the solution stated in the theorem qualifies indeed as an equilibrium.

4.3. Boundedness and recurrence

In the present subsection we collect a number of results dealing with the boundedness of capital and consumption paths in equilibrium. Almost all of these results have exact counterparts in the discrete-time setting, although the proofs in continuous time often require more elaborate arguments. We start by proving that, in every equilibrium, the aggregate capital stock as well as the individual capital holdings remain bounded.

Lemma 3. *There exists $\bar{K} > 0$ such that the inequalities $0 \leqslant K(t) \leqslant \bar{K}$ and $0 \leqslant k^h(t) \leqslant \bar{K}$ hold for all $h \in \mathcal{H}$ and all $t \in \mathbb{R}_+$. This is not only true for every equilibrium but for all feasible aggregate and individual capital paths.*

Proof. Because of (3) and (6) we have $\dot{K}(t) \leqslant f(K(t)) - \delta K(t)$. The properties of f and the assumption $\delta > 0$ ensure that the right-hand side of this inequality is non-positive for all sufficiently large $K(t)$, say, for all $K(t) \geqslant M$. Setting $\bar{K} = \max\{M, K_0\}$, where $K_0 = \sum_{h=1}^{H} k_0^h$, it follows that $0 \leqslant K(t) \leqslant \bar{K}$ holds for all $t \in \mathbb{R}_+$. The statement $0 \leqslant k^h(t) \leqslant \bar{K}$ follows then trivially from (2) and (5). □

We continue by showing that consumption also remains bounded. Whereas this result is rather trivial in discrete-time setting,[5] it requires some subtle arguments in the continuous-time framework. This is the case because consumption and investment (for each household) are flows in the continuous-time formulation, and there is no a priori upper bound on the choice of consumption, and no a priori bound on the choice of investment in the household's optimization problem. Thus, the upper bound on consumption (obtained in Lemma 4 below) results from using information beyond that available for feasible aggregate and individual capital paths.

Lemma 4. *There exists $\bar{c} > 0$ such that $0 \leqslant c^h(t) \leqslant \bar{c}$ holds for all $h \in \mathcal{H}$ and all $t \in \mathbb{R}_+$.*

[5] See Eq. (7) in Becker and Foias [4] and the sentence following that equation.

Proof. Non-negativity of $c^h(t)$ follows from (3). To demonstrate the existence of an upper bound on consumption, we proceed in four steps.

STEP 1: For all $j \in \mathcal{H}$ and all $t \in \mathbb{R}_+$ it holds that

$$\begin{aligned}[r(t)-\delta]k^j(t)+w(t) &\leqslant f'\big(K(t)\big)K(t)+W\big(K(t)\big)\\ &= (1/H)f\big(K(t)\big)+\big[(H-1)/H\big]f'\big(K(t)\big)K(t).\end{aligned}$$

Because of $f'(K)K < f(K)$ and Lemma 3, this implies

$$\big[r(t)-\delta\big]k^j(t)+w(t) < f\big(K(t)\big) \leqslant f(\bar{K}). \tag{13}$$

From this property and conditions (1) and (3) we obtain for all $j \in \mathcal{H}$ and all $t \in \mathbb{R}_+$ that

$$\dot{k}^j(t) = \big[r(t)-\delta\big]k^j(t)+w(t)-c^j(t) < f(\bar{K}).$$

STEP 2: Suppose that there exist $h \in \mathcal{H}$ and $\bar{t} \in \mathbb{R}_+$ such that $c^h(\bar{t}) > Hf(\bar{K})$. Because c^h is continuous, the inequality $c^h(t) > Hf(\bar{K})$ must hold for all $t \in I$, where I is an open interval containing $\bar{t}$. Together with (1) and (13) we obtain for all $t \in I$ that

$$\dot{k}^h(t) = \big[r(t)-\delta\big]k^h(t)+w(t)-c^h(t) < (1-H)f(\bar{K}) < 0. \tag{14}$$

From (5), (14), and the result of step 1 it follows for all $t \in I$ that

$$\dot{K}(t) = \sum_{j=1}^{H}\dot{k}^j(t) < (H-1)f(\bar{K})+(1-H)f(\bar{K}) = 0$$

and, therefore,

$$\dot{r}(t) = f''\big(K(t)\big)\dot{K}(t) > 0. \tag{15}$$

Furthermore, because of (2) the inequality $\dot{k}^h(t) < 0$ can only hold if

$$k^h(t) > 0. \tag{16}$$

We have therefore proved that, in the case where $c^h(\bar{t}) > Hf(\bar{K})$ holds at some instant $\bar{t}$ and for some household $h \in \mathcal{H}$, there exists an open interval I containing $\bar{t}$ such that conditions (14)–(16) must be satisfied for all $t \in I$.

STEP 3: Now suppose that there exist $T \in \mathbb{R}_+$ and $h \in \mathcal{H}$ such that $c^h(t) > Hf(\bar{K})$ holds for all $t \geqslant T$. In this case we see from (14) that $k^h(t)$ must eventually become negative. Since this would contradict condition (2), it follows that $\liminf_{t\to+\infty} c^h(t) \leqslant Hf(\bar{K})$. If the lemma were not true, there would therefore exists $h \in \mathcal{H}$ for which $\limsup_{t\to+\infty} c^h(t) = +\infty$ and $\liminf_{t\to+\infty} c^h(t) \leqslant Hf(\bar{K})$. These two properties together imply that c^h attains infinitely many local maxima with values greater than $Hf(\bar{K})$.

STEP 4: From step 3 we know that in the case where there exists $h \in \mathcal{H}$ such that c^h is unbounded, there must exist $\bar{t} \in \mathbb{R}_+$ such that c^h attains a local maximum at $\bar{t}$ and such that $c^h(\bar{t}) > Hf(\bar{K})$. Because of (16) it follows that $\bar{t}$ cannot be a junction point, nor can it be contained in a boundary interval. This implies (by Lemma 1) that μ^h must be differentiable at $\bar{t}$. Furthermore, because of (8) and the fact that $\bar{t}$ is a local maximum of c^h it follows that $\bar{t}$ is a local minimum of μ^h. These properties imply that $\dot{\mu}^h(\bar{t}) = 0$. Because of (9), (11), and (16) we have $\dot{\mu}^h(t) = [\rho^h+\delta-r(t)]\mu^h(t)$ for all t close to $\bar{t}$. This shows (again by Lemma 1) that μ^h is actually twice differentiable with

$$\ddot{\mu}^h(\bar{t}) = [\rho^h + \delta - r(\bar{t})]\dot{\mu}^h(\bar{t}) - \dot{r}(\bar{t})\mu^h(\bar{t}).$$

Substituting $\dot{\mu}^h(\bar{t}) = 0$ and using (15) we therefore see that $\ddot{\mu}^h(\bar{t}) < 0$, which is a contradiction to $\bar{t}$ being a local minimum of μ^h. This completes the proof of the lemma. □

The above lemma has a number of consequences that we collect in the following corollary. Recall that we have defined the steady state values r^* and K^* by $f'(K^*) = r^* = \rho^1 + \delta$.

Corollary 1. (a) *For all $t \in \mathbb{R}_+$ and all $h \in \mathcal{H}$ it holds that $\mu^h(t) \geqslant (u^h)'(\bar{c}) > 0$.*

(b) *For all $T \in \mathbb{R}_+$ it holds that*

$$\liminf_{t\to+\infty} \int_T^t [r^* - r(s)]\,\mathrm{d}s > -\infty. \tag{17}$$

(c) *There exists $\kappa \in \mathbb{R}_+$ such that $|\dot{K}(t)| \leqslant \kappa$ holds for all $t \in \mathbb{R}_+$.*

Proof. Part (a) is obvious from (8) and Lemma 4. From (9)–(10) for household $h = 1$ we have $\dot{\mu}^1(t) \leqslant [\rho^1 + \delta - r(t)]\mu^1(t) = [r^* - r(t)]\mu^1(t)$ or, equivalently, $(\mathrm{d}/\mathrm{d}t)\ln \mu^1(t) \leqslant r^* - r(t)$. Together with part (a) this implies for all $(T, t) \in \mathbb{R}_+^2$ satisfying $T \leqslant t$ that

$$0 < (u^h)'(\bar{c}) \leqslant \mu^1(t) \leqslant \mu^1(T)e^{\int_T^t [r^* - r(s)]\,\mathrm{d}s}.$$

Part (b) of the corollary follows from this inequality by letting t approach $+\infty$.

It remains to prove part (c). Define $\kappa = \max\{\delta\bar{K} + H\bar{c}, f(\bar{K})\}$. From (6) it follows for all $t \in \mathbb{R}_+$ that

$$\dot{K}(t) = f(K(t)) - \delta K(t) - \sum_{h=1}^H c^h(t) \leqslant f(\bar{K}) \leqslant \kappa,$$

where we have also used (2), (3), (5), Lemma 3, and the fact that f is increasing on $\mathbb{R}_+$. In a similar way, we can use (5), (6), Lemmas 3 and 4, and the fact that f is increasing on $\mathbb{R}_+$ with $f(K) \geqslant f(0) = 0$ for all $K \in \mathbb{R}_+$ to obtain for all $t \in \mathbb{R}_+$ that

$$-\dot{K}(t) = -f(K(t)) + \delta K(t) + \sum_{h=1}^H c^h(t) \leqslant -f(0) + \delta\bar{K} + H\bar{c} \leqslant \kappa.$$

These results establish that $|\dot{K}(t)| \leqslant \kappa$ for all $t \in \mathbb{R}_+$. □

Part (c) of the above corollary states that the time derivative of the aggregate capital stock remains uniformly bounded. This result has obviously no counterpart in the discrete-time setting. The following lemma, on the other hand, corresponds to Lemma 1 in Becker and Foias [4].

Lemma 5. *In every equilibrium it holds that* $\limsup_{t\to+\infty} K(t) \geqslant K^*$.

Proof. Suppose to the contrary that $\limsup_{t\to+\infty} K(t) < K^*$. Then there exist $T \in \mathbb{R}_+$ and $\varepsilon > 0$ such that $r(s) \geqslant r^* + \varepsilon$ holds for all $s \geqslant T$. Obviously, this is a contradiction to Corollary 1(b) and the proof of the lemma is complete. □

Let us define the value $\underline{K}$ by the condition $f'(\underline{K}) = \rho^H + \delta$. With this definition we can prove the following lemma, which corresponds to Proposition 2 in Becker and Foias [4].

Lemma 6. *There exists $T \in \mathbb{R}_+$ such that $K(t) \geqslant \underline{K}$ holds for all $t \geqslant T$.*

Proof. STEP 1: Suppose to the contrary that there exists a sequence $(t_i)_{i=1}^{+\infty}$ with $\lim_{i\to+\infty} t_i = +\infty$ and $K(t_i) < \underline{K}$ for all $i \in \mathbb{N}$. Together with Lemma 5 this implies that K must attain infinitely many local minima with values smaller than $\underline{K}$. Let $\bar{t}$ be such a local minimum. From (7) and the monotonicity of W it follows that w also attains a local minimum at $\bar{t}$. Furthermore, it must hold that $\dot{K}(\bar{t}) = 0$ and $\dot{w}(\bar{t}) = W'(K(\bar{t}))\dot{K}(\bar{t}) = 0$. In the following steps 2–5 we discuss implications of these properties depending on whether $\bar{t}$ is contained in an interior interval, a boundary interval, or in the set of junction points. In step 6 we shall then construct a contradiction to $\bar{t}$ being a local minimum of K.

STEP 2: Let $\bar{t}$ be contained in an interior interval of household h's optimization problem. From $K(\bar{t}) < \underline{K}$ and the continuity of K and k^h, it follows that there exists an interval I containing $\bar{t}$ such that $r(t) > \rho^h + \delta$ and $k^h(t) > 0$ for all $t \in I$. From (9)–(11) it follows therefore that $\dot{\mu}^h(t) = [\rho^h + \delta - r(t)]\mu^h(t) < 0$ for all $t \in I$. Together with (8) this implies that $\dot{c}^h(\bar{t}) > 0$. For later use in step 6 let us define $\mathcal{J}(\bar{t}) = \{h \in \mathcal{H} \mid k^h(\bar{t}) > 0\} \neq \emptyset$ and $\eta = \sum_{h\in\mathcal{J}(\bar{t})} \dot{c}^h(\bar{t}) > 0$.

STEP 3: Suppose that $\bar{t}$ is in the interior of a boundary interval of household h's utility maximization problem. Then it must hold that $c^h(t) = w(t)$ locally around $\bar{t}$ and therefore it follows that $\dot{c}^h(\bar{t}) = \dot{w}(\bar{t}) = 0$.

STEP 4: Next suppose that $\bar{t}$ is an entry point or a contact point. Then there exists $\varepsilon > 0$ such that $k^h(t) > 0$ holds for all $t \in (\bar{t} - \varepsilon, \bar{t})$. Using the argument employed in step 2 we obtain that $\dot{c}^h(t) > 0$ for all $t \in (\bar{t} - \varepsilon, \bar{t})$ and it follows that c^h is strictly increasing to the left of $\bar{t}$. Because $c^h(\bar{t}) = w(\bar{t})$ must hold (see Lemma 2) and because $\bar{t}$ is a local minimum of w, it follows that $c^h(t) < c^h(\bar{t}) = w(\bar{t}) \leqslant w(t)$ for all $t \in (\bar{t} - \varepsilon, \bar{t})$. Obviously, this is a contradiction to Lemma 2, and it follows that $\bar{t}$ can be neither an entry point nor a contact point.

STEP 5: Finally, assume that $\bar{t}$ is an exit point. In this case there exists $\varepsilon > 0$ such that $c^h(t) = w(t)$ for all $t \in (\bar{t} - \varepsilon, \bar{t})$. Thus, the left-hand derivative of c^h at $\bar{t}$ must coincide with the left-hand derivative of w at $\bar{t}$, which we have shown in step 1 to be equal to 0.

STEP 6: Steps 2–5 imply that the left-hand derivative of the function $t \mapsto -\sum_{h=1}^{H} c^h(t)$ at $t = \bar{t}$ exists and is given by $-\eta < 0$. Together with $\dot{K}(\bar{t}) = 0$ and (6) this implies that the left-hand derivative of

$$\dot{K}(t) = f\bigl(K(t)\bigr) - \delta K(t) - \sum_{h=1}^{H} c^h(t)$$

at $t = \bar{t}$ is negative. Hence, $\dot{K}$ must be strictly decreasing on an interval $(\bar{t} - \varepsilon, \bar{t})$ for some $\varepsilon > 0$. This, in turn, implies that $\dot{K}(t) > \dot{K}(\bar{t}) = 0$ for all $t \in (\bar{t} - \varepsilon, \bar{t})$ and it follows that K is strictly increasing immediately to the left of $\bar{t}$. This is a contradiction to $\bar{t}$ being a local minimum of K. □

Lemma 6 has the implication that the consumption level of any household $h \in \mathcal{H}$ does not converge to 0. This is the content of the following lemma which is similar to Corollary 1 in Becker and Foias [4].

Lemma 7. *For all $h \in \mathcal{H}$ it holds that* $\limsup_{t\to+\infty} c^h(t) > 0$.

Proof. From Lemma 6 it follows that there exists $T \in \mathbb{R}_+$ such that for all $t \geqslant T$ it holds that $w(t) = W(K(t)) \geqslant W(\underline{K}) > 0$. If the present lemma were not true, there would exist $h \in \mathcal{H}$ such that $\lim_{t\to+\infty} c^h(t) = 0$. But this would imply that there is $\bar{t} > T$ such that $c^h(t) < w(t)$ for all $t \geqslant \bar{t}$. Clearly we can choose $\bar{t}$ so that k^h is differentiable at $\bar{t}$ and c^h is continuous at $\bar{t}$. Define an alternative path $(\mathbf{k}^h, \mathbf{c}^h)$ as follows:

$$\mathbf{k}^h(t) = \begin{cases} k^h(t) & \text{for } t \in [0, \bar{t}\,], \\ k^h(\bar{t})e^{-\delta(t-\bar{t})} & \text{for } t > \bar{t}, \end{cases}$$

and

$$\mathbf{c}^h(t) = \begin{cases} c^h(t) & \text{for } t \in [0, \bar{t}\,], \\ w(t) + r(t)k^h(\bar{t})e^{-\delta(t-\bar{t})} & \text{for } t > \bar{t}. \end{cases}$$

Note that $\mathbf{k}^h$ is clearly continuous on $\mathbb{R}_+$ and piecewise differentiable. Further,

$$\dot{\mathbf{k}}^h(t) = \begin{cases} \dot{k}^h(t) = [r(t) - \delta]k^h(t) + w(t) - c^h(t) & \text{for } t \in [0, \bar{t}\,), \\ -\delta k^h(\bar{t})e^{-\delta(t-\bar{t})} = -\delta \mathbf{k}^h(t) & \text{for } t > \bar{t} \end{cases} \tag{18}$$

so that

$$\lim_{t\to\bar{t}-} \dot{\mathbf{k}}^h(t) = \big[r(\bar{t}) - \delta\big]k^h(\bar{t}) + w(\bar{t}) - c^h(\bar{t}) > -\delta k^h(\bar{t}) = \lim_{t\to\bar{t}+} \dot{\mathbf{k}}^h(t).$$

This shows that $\mathbf{k}^h$ is not differentiable at $t = \bar{t}$.

Note that $\mathbf{c}^h$ is piecewise continuous with finite left-hand and right-hand limits. Also, for all $t > \bar{t}$,

$$\mathbf{c}^h(t) = w(t) + r(t)k^h(\bar{t})e^{-\delta(t-\bar{t})} = w(t) + r(t)\mathbf{k}^h(t) \geqslant w(t) > c^h(t). \tag{19}$$

Finally, we show that $(\mathbf{k}^h, \mathbf{c}^h)$ satisfies the flow budget constraint (1) for all $t \in \mathbb{R}_+$ at which $\mathbf{k}^h$ is differentiable. Indeed, for $t \in [0, \bar{t})$, the points of differentiability of $\mathbf{k}^h$ are precisely the points of differentiability of k^h and we have

$$\dot{\mathbf{k}}^h(t) = \dot{k}^h(t) = \big[r(t) - \delta\big]k^h(t) + w(t) - c^h(t) = \big[r(t) - \delta\big]\mathbf{k}^h(t) + w(t) - \mathbf{c}^h(t).$$

For all $t > T$ we can use (18) and (19) to obtain

$$\dot{\mathbf{k}}^h(t) = -\delta \mathbf{k}^h(t) = \big[r(t) - \delta\big]\mathbf{k}^h(t) - r(t)\mathbf{k}^h(t) = \big[r(t) - \delta\big]\mathbf{k}^h(t) + w(t) - \mathbf{c}^h(t).$$

Since $(\mathbf{k}^h, \mathbf{c}^h)$ is feasible for the optimization problem of household h, (19) contradicts the fact that (k^h, c^h) solves this problem. This contradiction proves the lemma. □

We conclude this section by proving that the recurrence property holds for all equilibria. This result corresponds to Proposition 3 in Becker and Foias [4].

Lemma 8. *Every equilibrium satisfies the recurrence property, that is, for every household $h \geqslant 2$ there exists a sequence of time instants $(t_i^h)_{i=1}^{+\infty}$ with $\lim_{i\to+\infty} t_i^h = +\infty$ such that $k^h(t_i^h) = 0$ holds for all $i \in \mathbb{N}$.*

Proof. Suppose to the contrary that there exist $T \in \mathbb{R}_+$ and $h \geqslant 2$ such that $k^h(t) > 0$ for all $t \geqslant T$. Because of (9)–(11) this implies $\dot{\mu}^h(t) = [\rho^h + \delta - r(t)]\mu^h(t)$ for all $t \geqslant T$ and therefore

$$\mu^h(t) = \mu^h(T)e^{\int_T^t [\rho^h + \delta - r(s)]\,\mathrm{d}s} = \mu^h(T)e^{(\rho^h - \rho^1)(t-T)}e^{\int_T^t [r^* - r(s)]\,\mathrm{d}s}.$$

Because of Corollary 1(b) and $\rho^1 < \rho^h$ it follows that the right-hand side approaches $+\infty$ as t goes to $+\infty$. This, in turn, implies that $\lim_{t\to+\infty} \mu^h(t) = +\infty$ and it follows from (8) that $\lim_{t\to+\infty} c^h(t) = 0$. Because this contradicts Lemma 7 the proof of the present lemma is complete. □

4.4. Aggregate dynamics and the proof of Theorem 4

The present subsection contains results about the dynamics of the aggregate capital stock in the continuous-time Ramsey model. These results either have no counterparts at all in the Ramsey model in discrete time or they hold in discrete time only under additional assumptions on the production function.

We start by proving that the turnpike property implies convergence to the unique steady state equilibrium. It is known from various examples in the literature that there does not exist a corresponding result in the discrete-time formulation of the model; see, e.g., Becker and Foias [4,5], Sorger [16] or [17].

Lemma 9. *If an equilibrium satisfies the turnpike property, then it converges to the unique steady state equilibrium.*

Proof. Suppose that the turnpike property holds, that is, there exists $T \in \mathbb{R}_+$ such that $k^1(t) = K(t)$ and $k^h(t) = 0$ for all $t \geqslant T$ and all $h \geqslant 2$. In this case the equilibrium dynamics after time T can be described by just two differential equations. The first one is the flow budget constraint of household $h = 1$, which can be written as

$$\dot{K}(t) = M\big(K(t)\big) - \delta K(t) - \psi\big(\mu^1(t)\big), \tag{20}$$

where $M(K) = f'(K)K + W(K) = (1/H)[f(K) + (H-1)f'(K)K]$ and where ψ is the inverse of $(u^1)'$. The second equation is household 1's adjoint equation

$$\dot{\mu}^1(t) = \big[\rho^1 + \delta - f'\big(K(t)\big)\big]\mu^1(t). \tag{21}$$

We first show that the system of differential equations (20)–(21) has a unique fixed point. Indeed, if $\dot{\mu}^1(t) = 0$, then it follows from (21) that either $\mu^1(t) = 0$ or $f'(K(t)) = \rho^1 + \delta = r^*$. The former cannot hold since $\mu^1(t) = (u^1)'(c^1(t)) > 0$. Hence, we must have $K(t) = K^*$. Substituting this into (20) we obtain $c^1(t) = \psi(\mu^1(t)) = M(K^*) - \delta K^*$, which coincides with the corresponding value in the steady state equilibrium.

The Jacobian matrix of system (20)–(21) evaluated at the steady state is given by

$$\begin{pmatrix} M'(K^*) - \delta & -\psi'(\mu^1) \\ -f''(K^*)\mu^1 & 0 \end{pmatrix}.$$

Because ψ is the inverse of $(u^1)'$, it follows that $\psi'(\mu^1) < 0$ which, together with $\mu^1 > 0$ and $f''(K^*) < 0$, implies that the determinant of the Jacobian matrix is negative. This proves that the fixed point is a saddle point with one positive and one negative real eigenvalue.

We know from the results in Subsection 4.3 that every equilibrium satisfying the turnpike property corresponds to a bounded solution of system (20)–(21) (after some finite time T). Because the only fixed point of that system is a saddle point, there cannot exist any periodic orbits. This is an implication of index theory; see, e.g., Section 6.8 in Strogatz [18]. It follows therefore from the Poincaré–Bendixson theorem that every solution of (20)–(21) that corresponds to an equilibrium must converge to the unique fixed point; see Section 7.3 in Strogatz [18]. This implies that $\lim_{t\to+\infty} K(t) = K^*$, $\lim_{t\to+\infty} r(t) = f'(K^*)$, $\lim_{t\to+\infty} w(t) = w^*$, and $\lim_{t\to+\infty} c^1(t) = \lim_{t\to+\infty} \psi(\mu^1(t)) = M(K^*) - \delta K^*$. Since the turnpike property is assumed to hold, the capital holdings and consumption rates of all households $h \geqslant 2$ also converge to the corresponding steady state values. □

The following lemma is the key to all the results in the rest of this subsection. In the discrete-time framework, this result has only been proved under the so-called maximal income monotonicity assumption; see Lemma 2 in Becker et al. [3]. In the continuous-time model it holds under standard assumptions.

Lemma 10. *The aggregate capital stock K does not attain a local maximum at any $\bar{t} \in \mathbb{R}_+$ for which $K(\bar{t}) > K^*$.*

Proof. STEP 1: Suppose to the contrary that there exists $\bar{t} > 0$ such that $K(\bar{t}) > K^*$ and such that $\bar{t}$ is a local maximum of K. Then it follows that $\dot{K}(\bar{t}) = 0$ and that w attains a local maximum at $\bar{t}$. In the following steps 2–5 we shall show that there exists $\varepsilon > 0$ such that $c^h(t) \leqslant c^h(\bar{t})$ holds for all $h \in \mathcal{H}$ and for all $t \in (\bar{t}, \bar{t} + \varepsilon)$, and $\dot{c}^h(\bar{t}) < 0$ for all $h \in \mathcal{J}(\bar{t}) = \{j \in \mathcal{H} \mid k^j(\bar{t}) > 0\} \neq \emptyset$. In step 6 we shall derive a contradiction to $\bar{t}$ being a local maximum of K.

STEP 2: Let $h \in \mathcal{J}(\bar{t})$. From $K(\bar{t}) > K^*$ and the continuity of K and k^h it follows that there exists an interval I containing $\bar{t}$ such that $r(t) < \rho^h + \delta$ and $k^h(t) > 0$ for all $t \in I$. From (9)–(11) it follows therefore that $\dot{\mu}^h(t) = [\rho^h + \delta - r(t)]\mu^h(t) > 0$ for all $t \in I$. Together with (8) this implies that $\dot{c}^h(\bar{t}) < 0$. This implies of course that there exists $\varepsilon > 0$ such that $c^h(t) < c^h(\bar{t})$ holds for all $t \in (\bar{t}, \bar{t} + \varepsilon)$. For later use in step 6 let us define $\eta = \sum_{h \in \mathcal{J}(\bar{t})} \dot{c}^h(\bar{t}) < 0$.

STEP 3: Suppose that $\bar{t}$ is in the interior of a boundary interval of household h's utility maximization problem. Then it must hold that $c^h(t) = w(t)$ locally around $\bar{t}$ and it follows therefore that c^h has a local maximum at $\bar{t}$. This implies that there exists $\varepsilon > 0$ such that $c^h(t) \leqslant c^h(\bar{t})$ holds for all $t \in (\bar{t}, \bar{t} + \varepsilon)$.

STEP 4: Next suppose that $\bar{t}$ is an exit point or a contact point. Then there exists $\varepsilon > 0$ such that $k^h(t) > 0$ holds for all $t \in (\bar{t}, \bar{t} + \varepsilon)$. Using the argument employed in step 2 we obtain that $\dot{c}^h(t) < 0$ for all $t \in (\bar{t}, \bar{t} + \varepsilon)$ and it follows that c^h is strictly decreasing to the right of $\bar{t}$ and, hence, that $c^h(t) < c^h(\bar{t})$ holds for all $t \in (\bar{t}, \bar{t} + \varepsilon)$.

STEP 5: Finally, assume that $\bar{t}$ is an entry point. In this case there exists $\varepsilon > 0$ such that $c^h(t) = w(t) \leqslant w(\bar{t}) = c^h(\bar{t})$ for all $t \in (\bar{t}, \bar{t} + \varepsilon)$, where we have used the fact that $\bar{t}$ is a local maximum of w and that $c^h(\bar{t}) = w(\bar{t})$ holds at every junction point (see Lemma 2).

STEP 6: From the steps 2–5 it follows that there exists $\varepsilon > 0$ such that the function $t \mapsto -\sum_{h=1}^H c^h(t)$ is bounded below on $(\bar{t}, \bar{t} + \varepsilon)$ by the linearly increasing function

$$\sum_{h=1}^H \bigl[-c^h(\bar{t})\bigr] - (\eta/2)(t - \bar{t}).$$

Because $\dot{K}(\bar{t}) = 0$ it follows furthermore that the slope of the function $t \mapsto f(K(t)) - \delta K(t)$ at $t = \bar{t}$ is equal to 0. Putting these observations together and using (6) we can see that for all $t \in (\bar{t}, \bar{t} + \varepsilon)$ it holds that

$$\dot{K}(t) = f\bigl(K(t)\bigr) - \delta K(t) - \sum_{h=1}^H c^h(t) > \dot{K}(\bar{t}) = 0.$$

Hence, $\dot{K}(t) > 0$ holds immediately to the right of $\bar{t}$ which constitutes a contradiction to $\bar{t}$ being a local maximum of K. □

We can now derive the following important result; see also Lemma 3 in Borissov [8] for an analogous result in a discrete-time variant of the model that assumes that wages are paid before production takes place.

Corollary 2. *There exists $T \in \mathbb{R}$ such that one of the following two statements is correct*:

(a) *It holds for all $t \geqslant T$ that $K(t) \geqslant K^*$ and that K is monotonic on $[T, +\infty)$.*
(b) *It holds for all $t \geqslant T$ that $K(t) \leqslant K^*$.*

Proof. Suppose that there exists $\bar{t} \in \mathbb{R}_+$ such that $K(t) \geqslant K^*$ holds for all $t \in [\bar{t}, +\infty)$. In this case it follows from Lemma 10 that K can change monotonicity at most once on $[\bar{t}, +\infty)$. If it does not change monotonicity at all on $[\bar{t}, +\infty)$, then statement (a) holds with $T = \bar{t}$. If it changes monotonicity once at $T > \bar{t}$, then statement (a) holds as well.

Now suppose that there exists $\bar{t} \in \mathbb{R}_+$ such that $K(\bar{t}) < K^*$. If, in addition, it holds that $K(t) \leqslant K^*$ for all $t \geqslant \bar{t}$, then statement (b) is true with $T = \bar{t}$. Otherwise, there must exist $t_1 > \bar{t}$ such that $K(t_1) > K^*$. Note that in this case there cannot exist $t \geqslant t_1$ such that $K(t) \leqslant K^*$, because that would imply that K attains a local maximum at some $s \in [\bar{t}, t]$ and that $K(s) > K^*$. Since this would contradict Lemma 10, it follows that statement (a) must be true with some $T \geqslant t_1$. □

Using the above corollary we can now show that K converges.

Lemma 11. *It holds that $\lim_{t \to +\infty} K(t)$ exists. In the case described in Corollary 2(a), this limit must be greater than or equal to K^*; in the case described in Corollary 2(b), it must be equal to K^*.*

Proof. Consider first the situation described in part (a) of Corollary 2. Since K is monotonic on $[T, +\infty)$ and $K(t) \in [K^*, \bar{K}]$ holds for all $t \in [T, +\infty)$, it follows immediately that $\lim_{t \to +\infty} K(t)$ exists and that $\lim_{t \to +\infty} K(t) \geqslant K^*$.

Now consider the situation described in part (b) of Corollary 2, that is, there exists $T \in \mathbb{R}_+$ such that $K(t) \leqslant K^*$ holds for all $t \geqslant T$. Clearly, we can assume without loss of generality, that T is such that $K(t) \geqslant \underline{K}$ holds for all $t \geqslant T$ (using Lemma 6), where $\underline{K}$ is defined by the equation $f'(\underline{K}) = \rho^H + \delta$. The fact that $K(t) \leqslant K^*$ holds for all $t \geqslant T$ implies $r(t) \geqslant r^*$ for all $t \geqslant T$. From Lemma 5 we know that $\limsup_{t \to +\infty} K(t) \geqslant K^*$. If the present lemma is not true, then it must hold that $\liminf_{t \to +\infty} K(t) < K^*$. This implies that $\limsup_{t \to +\infty} r(t) > r^*$. This, in turn, implies that there exist $\varepsilon > 0$ and a sequence $(t_i)_{i=1}^{+\infty}$ with $\lim_{i \to +\infty} t_i = +\infty$ and $r(t_i) \geqslant r^* + \varepsilon$ for all $i \in \mathbb{N}$. Continuous differentiability of f on $\mathbb{R}_{++}$ implies that f' is uniformly continuous on the closed interval $[\underline{K}, \bar{K}]$. Condition (7) and Corollary 1(c) imply therefore that there exists $\sigma > 0$ such that $r(t) \geqslant r^* + \varepsilon/2$ for all $t \in [t_i - \sigma, t_i + \sigma]$ and all $i \in \mathbb{N}$. All of these facts together show that

$$\lim_{t \to +\infty} \int_T^t \left[r^* - r(s)\right] \mathrm{d}s = -\infty.$$

Since this contradicts Corollary 1(b), we obtain $\liminf_{t \to +\infty} K(t) = K^*$ and, consequently, $\lim_{t \to +\infty} K(t) = K^*$. □

To establish convergence of the aggregate capital stock towards K^* also in the case described in Corollary 2(a) we need two more results.

Lemma 12. *Assume that there exists $T \in \mathbb{R}$ such that $K(t) \geqslant K^*$ holds for all $t \geqslant T$ and such that K is monotonic on $[T, +\infty)$. Then it follows that K is non-increasing on $[T, +\infty)$ and that $\lim_{t \to +\infty} \dot{K}(t) = 0$.*

Proof. STEP 1: We first prove that K is non-increasing on $[T, +\infty)$. We distinguish two cases: the equilibrium satisfies the turnpike property or it does not. In the first case, we know from Lemma 9 that $\lim_{t\to+\infty} K(t) = K^*$. Since $K(t) \geqslant K^*$ holds for all $t \geqslant T$ and since K is monotonic on $[T, +\infty)$ by assumption, it must be the case that K is non-increasing on $[T, +\infty)$.

Now let us suppose that the turnpike property does not hold. In this case there exists a household $h \geqslant 2$ for which the equilibrium contains an interior interval $I = (t_1, t_2)$ with $T \leqslant t_1 < t_2$. Without loss of generality we may assume that t_1 is an exit point or a contact point and that t_2 is an entry point or a contact point. Because of the recurrence property from Lemma 8, t_2 must be finite. To summarize, we have $k^h(t_1) = k^h(t_2) = 0$ and $k^h(t) > 0$ for all $t \in I$. Since $K(t) \geqslant K^*$, we have $r(t) \leqslant r^* < \rho^h + \delta$, and since $k^h(t) > 0$ we have $\dot{\mu}^h(t) = [\rho^h + \delta - r(t)]\mu^h(t) > 0$ and, consequently, $\dot{c}^h(t) < 0$ for all $t \in I$. The function c^h is therefore strictly decreasing on I. Moreover, because t_1 and t_2 are junction points, we know from Lemma 2 that $w(t_1) = c^h(t_1)$ and $w(t_2) = c^h(t_2)$. Hence, it follows that $w(t_1) = c^h(t_1) > c^h(t_2) = w(t_2)$. Because we know that K is monotonic on $[T, +\infty)$, it follows that w must be monotonic on $[T, +\infty)$ as well. Because $w(t_1) > w(t_2)$ we see that w must be non-increasing. This, in turn, implies that K is non-increasing.

STEP 2: Next we prove that $\lim_{t\to+\infty} \dot{K}(t) = 0$. This is trivially true, if there exists $\bar{t} > T$ such that $K(\bar{t}) = K^*$, because then K must remain constant from $\bar{t}$ onwards. We may therefore assume that $K(t) > K^*$ holds for all $t \geqslant T$. Note that we must have $\dot{w}(t) = W'(K(t))\dot{K}(t) \leqslant 0$ (due to step 1) and $r(t) < \rho^h + \delta$ for all $t \geqslant T$ and all $h \in \mathcal{H}$. We claim that c^h is non-increasing on $[T, +\infty)$ for all $h \in \mathcal{H}$. Because c^h is continuous due to Lemma 1 and because junction points are isolated, it suffices to prove that c^h is non-increasing on boundary intervals and on interior intervals. On a boundary interval it holds that $c^h(t) = w(t)$ and we know already that w is non-increasing. On an interior interval we can use the argument from step 1 to show that c^h is strictly decreasing. Since c^h is non-increasing and non-negative on $[T, +\infty)$ for all $h \in \mathcal{H}$, it follows that $\lim_{t\to+\infty} c^h(t)$ must exist for all $h \in \mathcal{H}$. Using this result as well as the convergence of K (see Lemma 11) it follows from the output market clearing condition (6) that $\lim_{t\to+\infty} \dot{K}(t)$ must exist as well. This limit can obviously not differ from 0 because that would contradict the convergence of $K(t)$. $\square$

Lemma 13. *Suppose that* $\lim_{t\to+\infty} \dot{K}(t)$ *exists. Then it follows that the turnpike property holds.*

Proof. From Lemma 11 we know that $\lim_{t\to+\infty} K(t)$ exists. This implies obviously that $\lim_{t\to+\infty} \dot{K}(t)$, which is assumed to exist, must be equal to 0. From Lemma 11 we also know that $\lim_{t\to+\infty} K(t) \geqslant K^*$. Hence, there exist $T \in \mathbb{R}_+$ and $\varepsilon > 0$ such that $r(t) \leqslant \rho^h + \delta - \varepsilon$ for all $t \geqslant T$ and all $h \geqslant 2$. All of these observations together imply that there exists $\bar{t} \geqslant T$ such that

$$\frac{(u^h)''(w(t))}{(u^h)'(w(t))}\dot{w}(t) < \varepsilon \leqslant \rho^h + \delta - r(t) \tag{22}$$

holds for all $t \geqslant \bar{t}$ and all $h \geqslant 2$. This is the case because the two limits $\lim_{t\to+\infty} w(t) = \lim_{t\to+\infty} W(K(t)) \geqslant W(K^*) > 0$ and $\lim_{t\to+\infty} \dot{w}(t) = \lim_{t\to+\infty} W'(K(t))\dot{K}(t) = 0$ exist and u^h is twice continuously differentiable on $\mathbb{R}_{++}$.

Now suppose that the turnpike property does not hold. Then there exists a household $h \geqslant 2$ for which the equilibrium contains an interior interval $I = (t_1, t_2)$ with $\bar{t} \leqslant t_1 < t_2$. Without loss of generality we may assume that t_1 is an exit point or a contact point and that t_2 is an entry point or a contact point. Because of Lemma 8, t_2 must be finite. To summarize, we have $k^h(t_1) = k^h(t_2) = 0$ and $k^h(t) > 0$ for all $t \in I$.

Let us define $z(t) = (u^h)'(w(t)) - \mu^h(t)$. Noting that both w and μ^h must be differentiable on I, that $\dot{\mu}^h(t) = [\rho^h + \delta - r(t)]\mu^h(t)$ must hold for all $t \in I$, and that $\mu^h(t) = (u^h)'(c^h(t))$ by Eq. (8), it follows that

$$\dot{z}(t) = \big(u^h\big)''\big(w(t)\big)\dot{w}(t) - \big[\rho^h + \delta - r(t)\big]\big(u^h\big)'\big(c^h(t)\big). \tag{23}$$

From Lemma 2 we know that there must exist $t_3 \in I$ such that $c^h(t_3) < w(t_3)$ and, hence, $z(t_3) = (u^h)'(w(t_3)) - (u^h)'(c^h(t_3)) < 0$. We claim that $z(t) < 0$ holds for all $t \in (t_3, t_2)$. If this were not the case, the graph of $(u^h)'(w(t))$ would have to intersect the graph of $\mu^h(t) = (u^h)'(c^h(t))$ at some point $t \in (t_3, t_2)$ from below. From (22) and (23), however, we see that

$$\dot{z}(t)\big|_{w(t)=c^h(t)} = \big(u^h\big)'\big(w(t)\big)\left\{\frac{(u^h)''(w(t))}{(u^h)'(w(t))}\dot{w}(t) - \big[\rho^h + \delta - r(t)\big]\right\} < 0, \tag{24}$$

which rules out such an intersection. Therefore, the claim that $c^h(t) < w(t)$ holds for all $t \in (t_3, t_2)$ is proved. Because (t_3, t_2) is contained in the interior interval $I = (t_1, t_2)$, it holds that $k^h(t_3) > 0$. Using these results and integrating Eq. (1) it follows therefore that

$$k^h(t_2) = e^{\int_{t_3}^{t_2}[r(t)-\delta]\,\mathrm{d}t}\left\{k(t_3) + \int_{t_3}^{t_2} e^{-\int_{t_3}^{t}[r(s)-\delta]\,\mathrm{d}s}\big[w(t) - c^h(t)\big]\,\mathrm{d}t\right\} > 0.$$

As this contradicts the property $k^h(t_2) = 0$ that we assumed above, the proof of the lemma is complete. □

We are now ready to prove the main result about the aggregate dynamics of Ramsey equilibria.

Lemma 14. *Along every equilibrium it holds that* $\lim_{t\to+\infty} r(t) = r^*$, $\lim_{t\to+\infty} w(t) = w^*$, *and* $\lim_{t\to+\infty} K(t) = K^*$.

Proof. From Corollary 2 it follows that there are two cases to consider. If the equilibrium satisfies $K(t) \leqslant K^*$ for all sufficiently large t, then the lemma follows immediately from Lemma 11 and from (7). If $K(t) \geqslant K^*$ holds for all sufficiently large t, then we obtain from Corollary 2(a) and Lemma 12 that K is eventually non-increasing and that $\lim_{t\to+\infty} \dot{K}(t) = 0$. Together with Lemma 13 this implies that the turnpike property holds. The present lemma follows then from Lemma 9. □

We conclude the subsection by proving Theorem 4. From Corollary 2 we already know that there exists $T \in \mathbb{R}_+$ such that either $K(t) \geqslant K^*$ or $K(t) \leqslant K^*$ holds for all $t \geqslant T$. In the former case it follows from Lemma 12 that K is non-increasing on $[T, +\infty)$. This completes the proof of Theorem 4.

4.5. Individual dynamics and the proofs of Theorems 2 and 3

Having established the convergence of all aggregate variables in the previous subsection, we now turn to the individual capital holdings and consumption rates. We proceed in two separate lemmas.

Lemma 15. *Along every equilibrium it holds that* $\lim_{t\to+\infty} k^1(t) = K^*$ *and* $\lim_{t\to+\infty} k^h(t) = 0$ *for all* $h \geqslant 2$.

Proof. Because of condition (5) and Lemma 14 it is sufficient to prove the statement about the impatient households $h \geqslant 2$. Suppose that this statement is not true. Then there exist $h \geqslant 2$ and $\theta > 0$ such that $\limsup_{t\to+\infty} k^h(t) = \theta$. We shall show in three steps that this leads to a contradiction.

STEP 1: Because of Lemma 14 there exists $\bar{t} \in \mathbb{R}_+$ such that for all $t \geqslant \bar{t}$ it holds that

$$\left|r(t) - r^*\right| \leqslant \min\{\rho^1/2, (\rho^h - \rho^1)/2\} \tag{25}$$

and

$$\left|w(t) - w^*\right| \leqslant \varepsilon := \theta\rho^1/16. \tag{26}$$

Furthermore, because of the recurrence property established in Lemma 8, there exists $T \geqslant \bar{t}$ such that $k^h(T) = 0$.

STEP 2: We claim that for all $s \geqslant T$ it holds that $c^h(s) \leqslant w^* + \varepsilon$, where ε is defined in (26). There are two cases to consider: $k^h(s) = 0$ and $k^h(s) > 0$. In the first case, it must hold that $\dot{k}^h(s) \geqslant 0$ in order not to violate (2). It follows therefore from (1) that $0 \leqslant \dot{k}^h(s) = w(s) - c^h(s)$. Combining this with (26) we obtain $c^h(s) \leqslant w(s) \leqslant w^* + \varepsilon$ and the claim is proved.

Now consider the second case, in which $k^h(s) > 0$. This implies that s is contained in an interior interval. Let (t_1, t_2) be this interval where, without loss of generality, t_1 is an exit point or a contact point and t_2 an entry point or a contact point. Because of $s \geqslant T$, $k^h(T) = 0$, $k^h(s) > 0$, and the continuity of k^h, it follows that $T \leqslant t_1 < s$, $k^h(t_1) = 0$, and $k^h(t) > 0$ for all $t \in (t_1, s]$. From (9)–(11) it follows therefore that $\dot{\mu}^h(t) = [\rho^h + \delta - r(t)]\mu^h(t)$ for all $t \in (t_1, s]$. This implies that

$$\begin{aligned}
\dot{\mu}^h(t) &= \left[\rho^h + \delta - r(t)\right]\mu^h(t) \\
&= \left[\rho^h - \rho^1 + r^* - r(t)\right]\mu^h(t) \\
&\geqslant \left[\rho^h - \rho^1 - (\rho^h - \rho^1)/2\right]\mu^h(t) \\
&= (\rho^h - \rho^1)\mu^h(t)/2 > 0,
\end{aligned}$$

where we have used $r^* = \rho^1 + \delta$ and (25). Thus, μ^h is strictly increasing on $(t_1, s]$ and it follows from (8) that c^h must be strictly decreasing on that interval. Together with the continuity of c^h (see Lemma 1) we therefore obtain $c^h(s) < c^h(t_1)$. We have already seen in the first case that $k^h(t_1) = 0$ and $t_1 \geqslant T$ imply that $c^h(t_1) \leqslant w^* + \varepsilon$ so that we must have $c^h(s) < c^h(t_1) \leqslant w^* + \varepsilon$. This proves the claim in the second case.

STEP 3: Since $\limsup_{t\to+\infty} k^h(t) = \theta > 0$ there exists $t_1 > T$ such that $k^h(t_1) \geqslant \theta/2$, and because of the recurrence property there exists $t_2 > t_1$ such that $k^h(t_2) = 0$. Since k^h is continuous, it must attain a maximum on the compact interval $[t_1, t_2]$. Suppose that this maximum is attained at t_3. Then we have $k^h(t_3) \geqslant k^h(t_1) \geqslant \theta/2 > 0 = k^h(t_2)$ and, therefore, $t_3 < t_2$. We obtain

$$\begin{aligned}
\dot{k}^h(t_3) &= \left[r(t_3) - r^* + r^* - \delta\right]k^h(t_3) + \left[w(t_3) - w^*\right] + \left[w^* - c^h(t_3)\right] \\
&\geqslant -(\rho^1/2)k^h(t_3) + \rho^1 k^h(t_3) - \varepsilon - \varepsilon \\
&\geqslant \theta\rho^1/4 - 2\varepsilon \\
&= \theta\rho^1/8 > 0,
\end{aligned}$$

where we have used (1) for the first line, the definition of r^*, conditions (25) and (26), and the result from step 2 for the second line, the fact that $k^h(t_3) \geqslant \theta/2$ for the third line, and the definition of ε from (26) for the last line. The above chain of inequalities therefore proves that

$\dot{k}^h(t_3) > 0$ which is obviously a contradiction to k^h attaining its maximum on $[t_1, t_2]$ at the point $t_3 < t_2$. This contradiction completes the proof of the lemma. □

Lemma 16. *In every equilibrium it holds that* $\lim_{t\to+\infty} c^1(t) = (r^* - \delta)K^* + w^*$ *and* $\lim_{t\to+\infty} c^h(t) = w^*$ *for all* $h \geqslant 2$.

Proof. STEP 1: In this step we prove $\lim_{t\to+\infty} c^1(t) = (r^* - \delta)K^* + w^*$. Consider the two cases described in Corollary 2. In case (a) it follows from Lemmas 12 and 13 that the turnpike property holds and the claim follows from Lemma 9.

Now consider the situation described in part (b) of Corollary 2. Because $K(t) \leqslant K^*$ for all $t \geqslant T$ it must hold that $r(t) \geqslant r^* = \rho^1 + \delta$ for all $t \geqslant T$. Because of $\lim_{t\to+\infty} k^1(t) = K^* > 0$ (see Lemma 15) there must exist $\bar{t} \geqslant T$ such that $k^1(t) > 0$ for all $t \geqslant \bar{t}$. Together with conditions (9)–(11) these properties imply that $\dot{\mu}^1(t) = [\rho^1 + \delta - r(t)]\mu^1(t) \leqslant 0$. It follows that μ^1 is non-increasing on $[\bar{t}, +\infty)$ and, due to (8), that $c^1(t)$ is non-decreasing on $[\bar{t}, +\infty)$. Since we know from Lemma 4 that c^1 remains uniformly bounded, the limit of $c^1(t)$ as t approaches infinity must exist. This property together with Lemmas 14 and 15 shows that for $h = 1$ all terms on the right-hand side of (1) converge. Consequently, $\lim_{t\to+\infty} \dot{k}^1(t)$ must also exist. However, because $k^1(t)$ converges, the only possible limit of $\dot{k}^1(t)$ is 0. Substituting all of this into (1) it follows that $\lim_{t\to+\infty} c^1(t) = (r^* - \delta)K^* + w^*$.

STEP 2: For the rest of the proof let us fix a household $h \geqslant 2$. We first claim that $\limsup_{t\to+\infty} c^h(t) \leqslant w^*$. If this is not the case, there exists $\theta > 0$ such that $\limsup_{t\to+\infty} c^h(t) = w^* + \theta$. Because of Lemma 14 there exists $\bar{t} \in \mathbb{R}_+$ such that for all $t \geqslant \bar{t}$ it holds that

$$\left|r(t) - r^*\right| \leqslant \left(\rho^h - \rho^1\right)/2$$

and

$$\left|w(t) - w^*\right| \leqslant \varepsilon := \theta/2. \tag{27}$$

Furthermore, because of Lemma 8 there exists $T \geqslant \bar{t}$ such that $k^h(T) = 0$. In exactly the same way as in step 2 of the proof of Lemma 15 one can now show that $c^h(s) \leqslant w^* + \varepsilon$ holds for all $s \geqslant T$. By the definition of ε in (27) this is a contradiction to $\limsup_{t\to+\infty} c^h(t) = w^* + \theta$. Hence, we have proved $\limsup_{t\to+\infty} c^h(t) \leqslant w^*$.

STEP 3: Next we prove that $\limsup_{t\to+\infty} c^h(t) \geqslant w^*$ for all $h \geqslant 2$. Suppose to the contrary that there exists $\theta > 0$ such that $\limsup_{t\to+\infty} c^h(t) = w^* - \theta$. This implies that there exists $\bar{t} \in \mathbb{R}_+$ such that

$$c^h(t) \leqslant w^* - \theta/2 \tag{28}$$

holds for all $t \geqslant \bar{t}$. Because of Lemma 14 and $r^* = \rho^1 + \delta > \delta$ one can choose $T \geqslant \bar{t}$ such that for all $t \geqslant T$ it holds that $[r(t) - \delta]k^h(t) \geqslant 0$ and $w(t) - w^* \geqslant -\theta/4$. Together with (1) and (28) this implies that

$$\dot{k}^h(t) \geqslant \left[w(t) - w^*\right] + \left[w^* - c^h(t)\right] \geqslant \theta/4 > 0$$

for all $t \geqslant T$. Obviously, this is a contradiction to the boundedness of k^h (see Lemma 3).

STEP 4: In this step we prove that $\liminf_{t\to+\infty} c^h(t) \geqslant w^*$. Suppose to the contrary that there exists $\theta > 0$ such that $\liminf_{t\to+\infty} c^h(t) = w^* - \theta$. Because of Lemma 14 and $r^* = \rho^1 + \delta > \delta$ one can choose $\bar{t} \in \mathbb{R}_+$ such that for all $t \geqslant \bar{t}$ it holds that

$$\left[r(t) - \delta\right]k^h(t) \geqslant 0, \qquad \left|r(t) - r^*\right| \leqslant \left(\rho^h - \rho^1\right)/2, \qquad w(t) \geqslant w^* - \theta/4. \tag{29}$$

Furthermore, because of $\liminf_{t\to+\infty} c^h(t) = w^* - \theta$ there exists $t_1 \geqslant \bar{t}$ such that $c^h(t_1) \leqslant w^* - (3/4)\theta$ and because of the result from step 3 there exists $t_2 > t_1$ such that $c^h(t_2) \geqslant w^* - \theta/2$. Now define $t_3 = \inf\{t \in [t_1, t_2] \mid c^h(t) \geqslant w^* - \theta/2\}$. Since $c^h(t_2) \geqslant w^* - \theta/2$, the infimum is taken over a non-empty set and is therefore well-defined. Since $c^h(t_1) \leqslant w^* - (3/4)\theta < w^* - \theta/2$, the infimum must be strictly larger than t_1. Thus, the interval $[t_1, t_3]$ is non-degenerate and

$$c^h(t) \leqslant w^* - \theta/2 \tag{30}$$

holds for all t in $[t_1, t_3]$. Combining (1), (29), and (30) it follows for all $t \in [t_1, t_3]$ that

$$\dot{k}^h(t) \geqslant \left[w(t) - w^*\right] + \left[w^* - c^h(t)\right] \geqslant \theta/4 > 0.$$

This, in turn, implies that $k^h(t) > 0$ for all $t \in (t_1, t_3]$. From conditions (9)–(10) and (29) we therefore obtain

$$\dot{\mu}^h(t) = \left[\rho^h + \delta - r(t)\right]\mu^h(t) = \left[\rho^h - \rho^1 + r^* - r(t)\right]\mu^h(t) \geqslant \left(\rho^h - \rho^1\right)\mu^h(t)/2 > 0.$$

Hence, μ^h is strictly increasing on $(t_1, t_3]$ and it follows from (8) that c^h is strictly decreasing on that interval. Using continuity of c^h as well as the results from above it follows that

$$w^* - \theta/2 = c^h(t_3) < c^h(t_1) \leqslant w^* - (3/4)\theta.$$

Obviously, this is a contradiction and our claim is proved.

STEP 5: From steps 2 and 4 it follows obviously that $\lim_{t\to+\infty} c^h(t) = w^*$. This completes the proof of the lemma. □

We are now ready to establish Theorems 2 and 3. Theorem 3 is an immediate implication of Lemmas 14, 15, and 16. To see that Theorem 2 holds, we distinguish again the two cases described in Corollary 2. In case (a) we know from Lemma 12 that $\lim_{t\to+\infty} \dot{K}(t)$ exists and it follows therefore from Lemma 13 that the turnpike property holds. In case (b) of Corollary 2 the limit of all variables on the right-hand side of (1) exists, which implies that $\lim_{t\to+\infty} \dot{k}^h(t)$ exists for all $h \in \mathcal{H}$. Using (5) it follows therefore that $\lim_{t\to+\infty} \dot{K}(t)$ exists and Theorem 2 follows again from Lemma 13.

4.6. Proof of Theorem 5

The proof follows the general idea put forward by Malinvaud [12]. Consider any equilibrium and denote by K and C the aggregate capital path and the aggregate consumption path in that equilibrium. Note that $r(t) = f'(K(t))$ must hold due to (7). Defining

$$p(t) = e^{\int_0^t [\delta - r(s)]\,\mathrm{d}s}$$

we obtain $p(t) > 0$ for all $t \in \mathbb{R}_+$. Moreover, because of Theorem 3 we know that $\lim_{t\to+\infty} r(t) = r^* = \rho^1 + \delta > \delta$, which implies that $\lim_{t\to+\infty} p(t) = 0$. Finally, the definition of p together with (7) implies that $p(t)[f'(K(t)) - \delta] + \dot{p}(t) = 0$ holds for all $t \in \mathbb{R}_+$. Due to the concavity of f this proves that

$$p(t)\left[f\left(K(t)\right) - \delta K(t)\right] + \dot{p}(t)K(t) \geqslant p(t)\left[f(x) - \delta x\right] + \dot{p}(t)x \tag{31}$$

holds for all $x \geqslant 0$.

Now suppose that the theorem is not true. Then there exists a feasible capital path $\tilde{K}$ emanating from K_0 such that the corresponding consumption path $\tilde{C}$ dominates the given consumption path C. Defining $g(K) = f(K) - \delta K$ we have

$$\begin{aligned}
&\int_0^T p(t)\bigl[C(t) - \tilde{C}(t)\bigr]\,\mathrm{d}t \\
&\quad = \int_0^T p(t)\bigl[g\bigl(K(t)\bigr) - \dot{K}(t) - g\bigl(\tilde{K}(t)\bigr) + \dot{\tilde{K}}(t)\bigr]\,\mathrm{d}t \\
&\quad = \int_0^T p(t)\bigl[g\bigl(K(t)\bigr) - g\bigl(\tilde{K}(t)\bigr)\bigr] + \dot{p}(t)\bigl[K(t) - \tilde{K}(t)\bigr]\,\mathrm{d}t - p(T)\bigl[K(T) - \tilde{K}(T)\bigr],
\end{aligned}$$

where we have used partial integration and the fact that both capital paths K and $\tilde{K}$ start from the same initial value K_0. Noting that K must remain bounded due to Lemma 3 and combining the above result with (31) and $\lim_{t\to+\infty} p(t) = 0$ it follows therefore that

$$\liminf_{T\to+\infty} \int_0^T p(t)\bigl[C(t) - \tilde{C}(t)\bigr]\,\mathrm{d}t \geqslant 0.$$

Since this contradicts the assumption that $\tilde{C}$ dominates C, the proof of the theorem is complete.

5. Concluding remarks

The purpose of the present paper was to analyze the Ramsey model in a continuous-time setting in order to see which of the results that have been derived in the discrete-time formulation carry over to the continuous-time model and which ones need to be modified. It turned out that the continuous-time formulation allows for a full confirmation of the "folk result" about the eventual capital ownership pattern (Ramsey's conjecture), for a considerably more accurate description of the equilibrium dynamics, for a verification of the global asymptotic stability of the unique steady state equilibrium, and for a proof of the efficiency of all equilibria. All of these properties need not be true in the discrete-time setting unless one imposes non-standard assumptions.

We do not claim that the continuous-time formulation is more appropriate than the discrete-time formulation or vice versa, neither for the Ramsey model considered in the present paper nor for most of the other models that are used in economic research. However, as the present study clearly demonstrates, the differences in the predictions of models formulated in the two settings can be significant. As a consequence, one has to be very careful with intuitive explanations that do not take into account the way in which time is modeled.

References

[1] R.A. Becker, On the long-run steady state in a simple dynamic model of equilibrium with heterogeneous households, Quart. J. Econ. 95 (1980) 375–382.

[2] R.A. Becker, Equilibrium dynamics with many agents, in: R.-A. Dana, C. Le Van, T. Mitra, K. Nishimura (Eds.), Handbook on Optimal Growth, Springer-Verlag, Heidelberg, 2006, pp. 385–442.

[3] R.A. Becker, R.S. Dubey, T. Mitra, On Ramsey equilibrium: capital ownership pattern and inefficiency, Discussion paper, 2012.
[4] R.A. Becker, C. Foias, A characterization of Ramsey equilibrium, J. Econ. Theory 41 (1987) 173–184.
[5] R.A. Becker, C. Foias, The local bifurcation of Ramsey equilibrium, Econ. Theory 4 (1994) 719–744.
[6] R.A. Becker, T. Mitra, Efficient Ramsey equilibria, Macroecon. Dynam. 16 (2012) 18–32.
[7] T. Bewley, An integration of equilibrium theory and turnpike theory, J. Math. Econ. 10 (1982) 233–267.
[8] K. Borissov, On equilibrium dynamics with many agents and wages paid ex ante, Working paper Ec-05/11, Department of Economics, European University at St. Petersburg, 2011.
[9] G. Feichtinger, R.F. Hartl, Optimale Kontrolle ökonomischer Prozesse, de Gruyter, Berlin, 1986.
[10] I. Fisher, The Rate of Interest, MacMillan Company, New York, 1907, reprinted by Garland Publishing, 1982.
[11] R.F. Hartl, S.P. Sethi, R.G. Vickson, A survey of the maximum principles for optimal control problems with state constraints, SIAM Rev. 37 (1995) 181–218.
[12] E. Malinvaud, Capital accumulation and efficient allocation of resources, Econometrica 21 (1953) 233–268.
[13] J. Rae, Statement of Some New Principles of Political Economy, Hilliard, Gray, and Co., Boston, 1834, reprinted by August M. Kelly Publishers, 1964.
[14] F.P. Ramsey, A mathematical theory of saving, Econ. J. 38 (1928) 543–559.
[15] A. Seierstad, K. Sydsæter, Optimal Control Theory with Economic Applications, North-Holland, Amsterdam, 1987.
[16] G. Sorger, On the structure of Ramsey equilibrium: cycles, indeterminacy, and sunspots, Econ. Theory 4 (1994) 745–764.
[17] G. Sorger, Chaotic Ramsey equilibrium, Int. J. Bifurc. Chaos 5 (1995) 373–380.
[18] S.H. Strogatz, Nonlinear Dynamics and Chaos, Perseus Books, 1994.